7.1.1 First, we compute $m(s)$ as follows.

$$m(s) = \sum_{\theta=1}^{3} \pi(\theta) f_\theta(s) = \begin{cases} \frac{1}{5}\frac{1}{2} + \frac{2}{5}\frac{1}{3} + \frac{2}{5}\frac{3}{4} = \frac{8}{15} & s = 1 \\ \frac{1}{5}\frac{1}{2} + \frac{2}{5}\frac{2}{3} + \frac{2}{5}\frac{1}{4} = \frac{7}{15} & s = 2 \end{cases}$$

The posterior distribution of θ is then given by

θ	1	2	3
$\pi(\theta \mid s = 1)$	3/16	1/4	9/16
$\pi(\theta \mid s = 2)$	3/14	4/7	3/14

7.1.2 Since the posterior distribution of θ is Beta$(n\bar{x} + \alpha, n(1 - \bar{x}) + \beta)$ we have that

$$\begin{aligned} & E(\theta \mid x_1, \ldots, x_n) \\ &= \int_0^1 \theta \frac{\Gamma(n + \alpha + \beta)}{\Gamma(n\bar{x} + \alpha)\Gamma(n(1 - \bar{x}) + \beta)} \theta^{n\bar{x}+\alpha-1} (1 - \theta)^{n(1-\bar{x})+\beta-1} \, d\theta \\ &= \frac{\Gamma(n + \alpha + \beta)}{\Gamma(n\bar{x} + \alpha)\Gamma(n(1 - \bar{x}) + \beta)} \int_0^1 \theta^{n\bar{x}+\alpha} (1 - \theta)^{n(1-\bar{x})+\beta-1} \, d\theta \\ &= \frac{\Gamma(n + \alpha + \beta)}{\Gamma(n\bar{x} + \alpha)\Gamma(n(1 - \bar{x}) + \beta)} \frac{\Gamma(n\bar{x} + \alpha + 1)\Gamma(n(1 - \bar{x}) + \beta)}{\Gamma(n + \alpha + \beta + 1)} = \frac{n\bar{x} + \alpha}{n + \alpha + \beta}. \end{aligned}$$

and

$$
\begin{aligned}
&E\left(\theta^2 \,|\, x_1, \ldots, x_n\right) \\
&= \int_0^1 \theta^2 \frac{\Gamma(n+\alpha+\beta)}{\Gamma(n\bar{x}+\alpha)\,\Gamma(n(1-\bar{x})+\beta)} \theta^{n\bar{x}+\alpha-1}(1-\theta)^{n(1-\bar{x})+\beta-1}\, d\theta \\
&= \frac{\Gamma(n+\alpha+\beta)}{\Gamma(n\bar{x}+\alpha)\,\Gamma(n(1-\bar{x})+\beta)} \int_0^1 \theta^{n\bar{x}+\alpha+1}(1-\theta)^{n(1-\bar{x})+\beta-1}\, d\theta \\
&= \frac{\Gamma(n+\alpha+\beta)}{\Gamma(n\bar{x}+\alpha)\,\Gamma(n(1-\bar{x})+\beta)} \frac{\Gamma(n\bar{x}+\alpha+2)\,\Gamma(n(1-\bar{x})+\beta)}{\Gamma(n+\alpha+\beta+2)} \\
&= \frac{(n\bar{x}+\alpha)(n\bar{x}+\alpha+1)}{(n+\alpha+\beta)(n+\alpha+\beta+1)},
\end{aligned}
$$

so

$$
\begin{aligned}
\operatorname{Var}\left(\theta \,|\, x_1, \ldots, x_n\right) &= \frac{(n\bar{x}+\alpha)(n\bar{x}+\alpha+1)}{(n+\alpha+\beta)(n+\alpha+\beta+1)} - \left(\frac{n\bar{x}+\alpha}{n+\alpha+\beta}\right)^2 \\
&= \frac{(n\bar{x}+\alpha)(n(1-\bar{x})+\beta)}{(n+\alpha+\beta)^2(n+\alpha+\beta+1)}.
\end{aligned}
$$

7.1.3 First, the prior distribution of θ is $N(0, 10)$, therefore, the prior probability that θ is positive is 0.5. Next, the posterior distribution of θ is

$$
N\left(\left(\frac{1}{10}+\frac{10}{1}\right)^{-1}\left(\frac{10}{1}\right), \left(\frac{1}{10}+\frac{10}{1}\right)^{-1}\right) = N\left(0.99010, 9.9010 \times 10^{-2}\right).
$$

Therefore, the posterior probability that $\theta > 0$ is $1 - \Phi\left((0 - 0.99010)/\sqrt{9.9010 \times 10^{-2}}\right) = 1 - \Phi(-3.1466) = 1 - 0.0008 = 0.9992.$

7.1.4 The likelihood function is given by $L(\lambda \,|\, x_1, \ldots x_n) = e^{-n\lambda}\lambda^{n\bar{x}}/\prod(x_i!)$. The prior distribution has density given by $\beta^\alpha \lambda^{\alpha-1} e^{-\beta\lambda}/\Gamma(\alpha)$. The posterior density of λ is then proportional to $\beta^\alpha \lambda^{n\bar{x}+a-1} e^{-\lambda(n+\beta)}/\Gamma(\alpha)\prod(x_i!)$, and we recognize this as being proportional to the density of a Gamma$(n\bar{x}+a, n+\beta)$ distribution.

7.1.5 The likelihood function is given by $L(\theta \,|\, x_1, \ldots x_n) = \frac{1}{\theta^n} I_{[x_{(n)}, \infty)}(\theta)$. The prior distribution is the same as in the previous exercise. The posterior distribution of θ is then given by

$$
\pi(\theta \,|\, x_1, \ldots x_n) \propto \theta^{\alpha-n-1} e^{-\beta\theta} I_{[x_{(n)}, \infty)}(\theta) \Big/ \int_{x_{(n)}}^{\infty} \theta^{\alpha-n-1} e^{-\beta\theta}\, d\theta.
$$

7.1.6 From Problem 3.2.23 the posterior mean of θ_i is

$$
\frac{f_i+\alpha_i}{f_1+\alpha_1+f_2+\alpha_2+f_3+\alpha_3} = \frac{f_i+\alpha_i}{n+\alpha_1+\alpha_2+\alpha_3}
$$

and the posterior variance of θ_i is given by

$$
\frac{(f_i+\alpha_i)(f_1+\alpha_1+f_2+\alpha_2+f_3+\alpha_3-f_i-\alpha_i)}{(n+\alpha_1+\alpha_2+\alpha_3)^2(n+\alpha_1+\alpha_2+\alpha_3+1)}.
$$